Brief Announcement: Direction-Incentivized Spectral Partitioning for Acyclic Graphs

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Abstract

Partitioning directed acyclic graphs is central to scheduling, pipelining, and memory-hierarchy optimization. We break the symmetry in classical spectral bi-partitioning in order to incentivize the alignment of directed graph cuts and find partitions whose cut edges mostly point in one direction. Our method employs a modified spectral quadratic objective function that promotes cut edges with consistent directionality. We further introduce an algorithm that rectifies misaligned edges to produce acyclic bi-partitions with low balanced cut metrics, and extend it to the retrieval of topological orders. We demonstrate the effectiveness of the proposed algorithms in comparative experiments on real-world cases that focus on acyclic density-based bi-partitioning.

CCS Concepts

• **Theory of computation** → *Graph algorithms analysis*; • **Mathematics of computing** → *Spectra of graphs*; *Graph algorithms.*

Keywords

Spectral partitioning, (nearly) acyclic partitioning, directed acyclic graph (DAG) partitioning, conductance

ACM Reference Format:
Dimosthenis Pasadakis, Raphael S. Steiner, Pál András Papp, Toni Böhnlein, and Albert-Jan N. Yzelman. 2026. Brief Announcement: Direction-Incentivized Spectral Partitioning for Acyclic Graphs. In *38th ACM Symposium on Parallelism in Algorithms and Architectures (SPAA '26), July 06–10, 2026, London, United Kingdom.* ACM, New York, NY, USA, 4 pages. https://doi.org/10.1145/3816782.3819184

1 Introduction

Graph partitioning is central for a variety of applications where the goal is to identify tightly knit groups of roughly the same size with few interconnects. A plethora of methods have been developed for this problem; we refer to the surveys [7, 8, 15]. A specialization of the graph partitioning problem is acyclic partitioning, where one is given a directed acyclic graph (DAG) and the partition itself[1] is required to form a DAG. This stricter problem is required, e.g., when the DAG models a computation with dependencies, and has been used for pipeline computations, efficient uses of the memory hierarchy, and offline scheduling heuristics [2, 12, 26, 28].

Between undirected and acyclic graph partitioning lies a more relaxed partitioning problem: given a directed graph, one seeks to find a partition such that between any two parts the cut edges mostly point in one direction. This type of partitioning can be used as initializers for acyclic partitions and for flow-based clustering [18], such as hierarchy classifications [36], interbank debts [1], and population migration patterns [37].

Like balanced undirected bi-partitioning, balanced acyclic bi-partitioning is NP-hard [16, 23]. A variety of heuristics have been introduced, ranging from topological-order-based [21], to acyclic adaptations of the Fiduccia–Mattheyses (FM) algorithm [10, 31], evolutionary [24, 32], and multi-level algorithms [20, 31]. We refer to [8] and references therein for further information.

In this paper, we present a modification to the classic spectral partitioning method of Fiedler [14], which yields such a nearly acyclic bi-partition. In the case of a DAG, we furthermore augment the nearly acyclic bi-partition to an acyclic bi-partition. To the best of our knowledge, this marks the first time spectral methods have been used to address the problem of acyclic partitioning. Nonetheless, we note that several nearly-acyclic flow [18, 22, 36] and density-based [17, 38] spectral partitioners exist in the literature.

Our code and data are publicly available at https://github.com/Algebraic-Programming/SpectralTopSort. A preprint of the full paper is available on arXiv [30].

2 Acyclic spectral bi-partitioning

Classical spectral bi-partitioning is a continuous relaxation of the discrete min-cut problem on an undirected graph $G = (V, E)$. The problem may be stated as

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[1] Specifically the quotient graph with self-loops removed.

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SPAA '26, London, United Kingdom
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ACM ISBN 979-8-4007-2761-0/2026/07
https://doi.org/10.1145/3816782.3819184

$$\underset{\substack{\boldsymbol{x}\in\mathbb{R}^V\\ \|\boldsymbol{x}\|_2=1\\ \boldsymbol{x}\perp\mathbf{1}}}{\operatorname{argmin}} \sum_{(u,v)\in E} (x_u - x_v)^2 \tag{1}$$

and its solution is given by the Fiedler eigenvector [9, 14].

In this paper, we consider a directed graph G and would like the cut edges to mostly align in the same direction. We achieve this by introducing a penalty to Eq. (1) when the cut edges are not aligned.

2.1 Symmetry-breaking quadratic form

Due to the symmetry of the quadratic form, we are unable to enforce that a cut edge $x_u - x_v \not\approx 0$, $(u,v) \in E$, is positive. However, by considering pairs of edges, we are able to construct a quadratic form which is large if and only if all cut edges point in the same direction; that is, they have the same sign:

$$\sum_{\substack{(u_1,v_1)\in E\\ (u_2,v_2)\in E}} (x_{u_1} - x_{v_1})(x_{u_2} - x_{v_2}) = \left(\sum_{(u,v)\in E} (x_u - x_v)\right)^2. \tag{2}$$

We arrive at the symmetric quadratic form

$$Q_c(\boldsymbol{x}) = \sum_{(u,v)\in E} (x_u - x_v)^2 - c\left(\sum_{(u,v)\in E} (x_u - x_v)\right)^2, \tag{3}$$

which is semi-positive definite for $0 \le c \le \frac{1}{|E|}$ by Cauchy–Schwarz. More precisely, we can squeeze the quadratic form (3) between multiples of the quadratic form (1) used in classic spectral partitioning.

$$\sum_{(u,v)\in E} (x_u - x_v)^2 \ge Q_c(\boldsymbol{x}) \ge (1 - c|E|) \sum_{(u,v)\in E} (x_u - x_v)^2. \tag{4}$$

For $0 \le c < \frac{1}{|E|}$, it follows that (3) is zero if and only if the Laplacian (1) is zero. Hence, like the Laplacian, the kernel of quadratic form (3) identifies the weakly connected components of the graph G. It furthermore follows that theoretical results on classic spectral partitioning on cut quality continue to hold with an additional multiplicative loss of at most $(1 - c|E|)^{-1}$, e.g., (undirected) Cheeger's inequality [3, 4, 35].

We are thus able to use Eq. (3) for spectral bi-partitioning. More precisely, for a fixed $0 \le c \le \frac{1}{|E|}$, we minimize $Q_c(\boldsymbol{x})$ over $\boldsymbol{x} \in \mathbb{R}^V$ with $\|\boldsymbol{x}\|_2 = 1$ and $\boldsymbol{x} \perp \mathbf{1}$. In order to get from the solution of the optimization problem to a bi-partition[2] $V = S \sqcup T$, we take the pointwise sign of the optimal vector. The direction-incentivized spectral bi-partitioning algorithm is presented in Algorithm 2.1.

2.2 Acyclic fix

The bi-partitions obtained by Algorithm 2.1 can be turned into fully acyclic ones with a heuristic post-processing approach. The idea we present here to rectify misaligned edges in a bi-partition of a directed acyclic graph is based on a topological order and a bisection thereof [21]. In order to preserve as much from the original bi-partition as possible, we generate a topological order via a priority, where priority is given to vertices of the first part over those in the second part. Furthermore, as a secondary priority,

[2]Throughout the paper, we use the notation $S \sqcup T$ to denote a disjoint union.

Algorithm 2.1: Direction-incentivised spectral bi-partition.

Data: A finite directed graph $G = (V, E)$.
Result: A small cut bi-partition $S \sqcup T = V$ such that most edges between S and T are from S to T.

$\boldsymbol{x} \leftarrow \boldsymbol{y} \in \mathbb{R}^V$ minimizing (3) over $\|\boldsymbol{y}\|_2 = 1$ and $\boldsymbol{y} \perp \mathbf{1}$
$S \leftarrow \{v \in V \mid x_v > 0\}$
$T \leftarrow \{v \in V \mid x_v \le 0\}$
if $|(S \times T) \cap E| < |(T \times S) \cap E|$ **then** $(S, T) \leftarrow (T, S)$
return (S, T)

vertices with a cut outgoing edge in the right direction are deprioritized whereas ones with a cut outgoing edge in the wrong direction are prioritized, and vice versa for cut incoming edges. In the case where the initial bi-partition came from the direction-incentivized spectral partitioning, see Algorithm 2.1, we are given more nuanced information via the minimization of Eq. (3). This could also be used as the first priority in the acyclic fix algorithm instead, by either using the values directly or discretizing them first. The details of our acyclic fix approach can be found in [30].

2.3 Extension to topological orders

Given a directed acyclic graph $G = (V, E)$ and a method to acyclicly bi-partition a DAG, one may apply said method recursively to obtain a topological order. To achieve this, one proceeds as follows. After applying the acyclic bi-partition ℓ-times, we are given an (acyclic) partition $\bigsqcup_{i=0}^{\ell} L_i = V$ of the set of vertices, such that there are no edges originating from L_j and ending in L_i for all $j > i$. We now take an index i such that $|L_i| \ge 2$ and acyclicly bi-partition $G|_{L_i}$ into S and T such that both sets are non-empty and there is no edge from T to S. We now re-index $L_{j+1} \leftarrow L_j$ for $j = \ell, \ell - 1, \ldots, i + 1$ and set $L_{i+1} \leftarrow T$ and $L_i \leftarrow S$. Once $|L_i| = 1$ for all i, we have obtained a valid topological order.

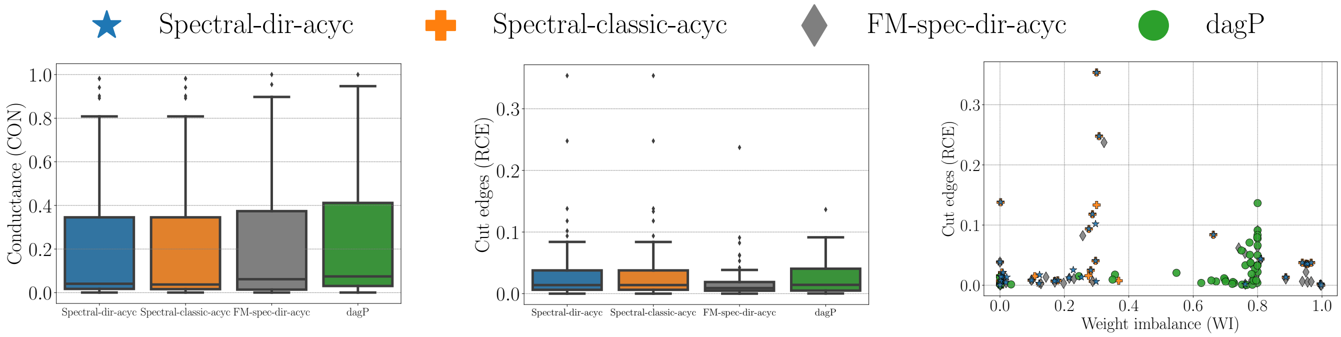
This scheme resembles the one in [33] for undirected graphs, but with the added difficulty of generating a linear layout which is also a topological order. In our work, we improve on the above by taking into account prior decisions. That is, instead of simply bi-partitioning $G|_{L_i}$, we bi-partition G using the method from §2.2 where we fix $x_v = \operatorname{sign}(j - i)/\sqrt{|V|}$ for $v \in L_j$ with $j \ne i$. This significantly improves the bandwidth and other locality metrics of the resulting topological order. The details of the spectral topological order algorithm can be found in [30].

3 Numerical results

In this section, we demonstrate that spectral partitioning methods in conjunction with the acyclic fix produce acyclic bi-partitions that are on par, in terms of various metrics, with the ones produced by state-of-the-art algorithms, and that this can be further improved by local-search algorithms.

3.1 Experimental setup

We compare our Alg. 2.1 with $c = \frac{1}{2|E|}$, together with the acyclic fix, with the classic spectral partitioning also combined with our acyclic fix, and with the following algorithms from the literature: (1) acyclicity-adapted FM [10, 31] with initial acyclic bi-partition



(A) (B) (C)

Figure 1: Results on obtaining acyclic bi-partitions from the set of real-world DAGs. (A): Box plot distribution of the CON results for each algorithm under consideration, (B): Box plot distribution of the RCE results for each algorithm under consideration, (C): Cut edges (RCE) over weight imbalance (WI).

given by our direction-incentivized spectral bi-partition, and (2) the acyclic partitioner dagP [19, 20]. [3] For each algorithm supporting weight-imbalance constraints, we set the maximum weight imbalance to 0.8 to avoid artificially restricting the search space.

Given a finite directed graph $G = (V, E)$ and bi-partition $V = S \sqcup T$ of its vertices, we consider the classic directed partitioning metrics conductance (CON), ratio of cut over total edges (RCE), and weight imbalance (WI) [15]. We consider a set of 36 real-world directed (RW) graphs from the university of Florida sparse matrix collection [11] and 4 fine-grained scheduling graphs generated by the HyperDAG_DB [27, 28]. Together, these graphs emerge from a diverse set of applications, ranging from chemical process simulations to electromagnetics and fluid dynamics, and have number of vertices in the range $[10^2, 1.6 \times 10^4]$ and number of edges in the range $[10^3, 10^6]$. To evaluate our acyclic fix we convert this set to acyclic instances. A complete list of the matrices, their associated statistics, and the acyclicity conversion approach is offered in [30].

3.2 Acyclic bi-partitioning

We present in Figure 1 the bi-partitioning results on the real-world directed acyclic graphs. The performance of the four considered algorithms is shown: our proposed Algorithm 2.1 (in blue, Spectral-dir-acyc), classic spectral bisection (in orange, Spectral-classic-acyc), the acyclicity-adapted Fiduccia–Mattheyses variant using Algorithm 2.1 as initializer (in gray, FM-spec-dir-acyc), and dagP (in green). The first three methods incorporate the acyclic fix outlined in §2.2. In Subfigures (A), (B), we show box plots of the CON and RCE distribution results respectively, and in (C) the trade off between RCE and WI as a scatter plot. For CON, the two spectral methods exhibit the lowest median values, with the FM-based variant and classic spectral most frequently achieving the best score (47.5% each). Regarding RCE, the FM-based method achieves the lowest median, with dagP obtaining the best score in the highest number of instances (60.0%). The scatter plot in Subfigure (C) illustrates that the performance of dagP on RCE is associated with high weight imbalance (WI), a metric for which the classic spectral method attains the best result in 57.5% of cases, followed by the direction-incentivized proposed variant in 50.0%. These results highlight the effectiveness of the FM-based refinement of spectral methods to retrieve bi-partitions with low conductance and number of cut edges.

[3]dagP [19, 20] is based on a multilevel hierarchical scheme. Its coarsening phase employs agglomerative clustering heuristics, designed to contract vertices without introducing cycles. The initial bisection at the coarsest level is retrieved with a greedy directed graph growing strategy, and is subsequently refined in the uncoarsening levels using a boundary FM [13] style algorithm, which is adapted for directed graphs to only consider vertex moves that preserve acyclicity.

4 Concluding remarks

We have introduced a direction-incentivised spectral method for nearly-acyclic partitioning, that with appropriate fixes, is viable for acyclic partitioning and for the retrieval of topological orders. We achieved this by incorporating a direction-alignment term into the symmetric positive semi-definite quadratic objective of spectral partitioning, which adds only negligible overhead, as the added term and its derivatives are computable in linear time. Additionally, the proposed acyclic and direction-fix algorithms are linear. This motivates us to pursue high-performance and parallel implementations [5, 25, 29] and non-linear spectral generalizations [6, 34] in future work.

Acknowledgments

We thank Davide B. Bartolini and Mehdi Alipour for stimulating conversations on this and neighboring topics. Dimosthenis Pasadakis was financially supported by the joint DFG-470857344 and SNSF-204817 project, and by the Huawei Research Center Zurich.

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Received 20 February 2007; revised 12 March 2009; accepted 5 June 2009