Derivation of LU factorisation of a matrix A

Inspired by Gaussian elimination

Gaussian elimination

- Gaussian elimination: transform $Ax=b$ to $Ux=y$ with U upper triangular
- Linear Algebra:

  there exists a lower triang. transformation matrix T such that $TA = U$
- Let $L = T^{-1}$ -> L is also lower triangular !
- A = T^{-1}

- *Note: shift in notation: row & column indices: 1 to n (instead of from 0 to n-1).*

$$\underbrace{\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline -a_{1,1}^{-1}a_{2,1} & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ -a_{1,1}^{-1}a_{n,1} & 0 & \cdots & 1 \end{array}\right]}_{T^{(1)}} \cdot \underbrace{\left[\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ \hline a_{2,1} & a_{2,2} & \cdots & 0 \\ \vdots & & & \vdots \\ a_{m,1} & a_{n,2} & \cdots & a_{n,n} \end{array}\right]}_{A} = \underbrace{\left[\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \hline 0 & a_{2,2}^{(1)} & \cdots & a_{2,n}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n,2}^{(1)} & \cdots & a_{n,n}^{(1)} \end{array}\right]}_{U^{(1)}}$$

$$\underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline -\ell^{(1)} & I \end{array}\right]}_{T^{(1)}} \cdot \underbrace{\left[\begin{array}{c|c} a_{1,1} & a_{1,2:m} \\ \hline a_{2:m,1} & a_{2:m,2:m} \end{array}\right]}_{A} = \underbrace{\left[\begin{array}{c|c} a_{1,1} & a_{1,2:m} \\ \hline 0 & a_{2:m,2:m} - \ell^{(1)} a_{1,2:m} \end{array}\right]}_{U^{(1)}}$$

$$L^{(1)} := \left(T^{(1)}\right)^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ \hline \ell^{(1)} & I \end{array}\right]$$

$$\underbrace{\left[\begin{array}{c|c} a_{1,1} & a_{1,2:m} \\ \hline a_{2:m,1} & a_{2:m,2:m} \end{array}\right]}_{A} = \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline \ell^{(1)} & I \end{array}\right]}_{L^{(1)}} \cdot \underbrace{\left[\begin{array}{c|c} a_{1,1} & a_{1,2:m} \\ \hline 0 & a^{(1)}_{2:m,2:m} \end{array}\right]}_{U^{(1)}}$$

$$a^{(1)}_{2:m,2:m} = a_{2:m,2:m} - \ell^{(1)} a_{1,2:m}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\ell^{(2)} & I \end{bmatrix}}_{T^{(2)}} \cdot \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3:m} \\ 0 & a^{(1)}_{2,2} & a^{(1)}_{2,2:m} \\ 0 & a^{(1)}_{3:m,2} & a^{(1)}_{3:m,3:m} \end{bmatrix}}_{U^{(1)}} = \underbrace{\begin{bmatrix} a_{1,1} & a_{12,2} & a_{1,3:m} \\ 0 & a_{2,2} & a^{(1)}_{2,3:m} \\ 0 & 0 & a^{(1)}_{3:m,3:m} - \ell^{(2)} a_{2,3:m} \end{bmatrix}}_{U^{(2)}}$$

$$\underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3:m} \\ 0 & a^{(1)}_{2,2} & a^{(1)}_{2,2:m} \\ 0 & a^{(1)}_{3:m,2} & a^{(1)}_{3:m,3:m} \end{bmatrix}}_{U^{(1)}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ell^{(2)} & I \end{bmatrix}}_{L^{(2)}} \cdot \underbrace{\begin{bmatrix} a_{1,1} & a_{12,2} & a_{1,3:m} \\ 0 & a_{2,2} & a^{(1)}_{2,3:m} \\ 0 & 0 & a^{(2)}_{3:m,3:m} \end{bmatrix}}_{U^{(2)}}$$

$$a^{(2)}_{3:m,3:m} = a^{(1)}_{3:m,3:m} - \ell^{(2)} a_{2,3:m}$$

$$U^{(0)} = A$$

$$U^{(k)} = L^{(k+1)} U^{(k+1)} \quad , \quad k = 0, \ldots, k-1$$

$$U = U^{(n)}$$

$$L = L^{(1)} \ldots L^{(n)}$$

$$A = LU$$

$$L = L^{(1)} \cdots L^{(n)} = \begin{bmatrix} 1 & 0 & & & \\ \vdots & 1 & 0 & & \\ \vdots & \vdots & \ddots & & \\ \ell^{(1)} & \ell^{(2)} & & 1 & \\ \vdots & \vdots & & \ell^{(n-1)} & 1 \end{bmatrix}$$

Algorithm 2.1. Sequential LU decomposition.

input: $A^{(0)}$: $n \times n$ matrix.

output: L : $n \times n$ unit lower triangular matrix,
U : $n \times n$ upper triangular matrix,
such that $LU = A^{(0)}$.

for $k := 0$ **to** $n-1$ **do**
    for $j := k$ **to** $n-1$ **do**
        $u_{kj} := a_{kj}^{(k)}$;
    for $i := k+1$ **to** $n-1$ **do**
        $l_{ik} := a_{ik}^{(k)}/u_{kk}$;
    for $i := k+1$ **to** $n-1$ **do**
        for $j := k+1$ **to** $n-1$ **do**
            $a_{ij}^{(k+1)} := a_{ij}^{(k)} - l_{ik}u_{kj}$;