Graph matching for BSP

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Matching

Introduction

Greedy matching

BSP algorithm for edge-weighted matching

Sequential approximation algorithm

BSP approximation algorithm

Conclusion

Matchmaker, Matchmaker, Make me a match

From the film *Fiddler on the roof*

- Hodel: Well, somebody has to arrange the matches. Young people can't decide these things themselves.
- Hodel: For Papa, make him a scholar.
- Chava: For Mama, make him rich as a king.

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Matching can win you a Nobel prize

Marriage as an Economic Problem

Lloyd Shapley and Alvin Roth win the Nobel Prize for showing the best way to match people with what they really want.

By Matthew Yglesias | Posted Monday, Oct. 15, 2012, at 1:51 PM ET

65 SOCIAL OFF Like 963 Tweet 100

The Nobel Prize in economics went to Alvin E. Roth and Lloyd S. Shapley "for the theory of stable allocations and the practice of market design"

Source: Slate magazine October 15, 2012

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Motivation of graph matching

- Graph matching is a pairing of neighbouring vertices.
- It has applications in
  - medicine: finding suitable donors for organs
  - social networks: finding partners
  - scientific computing: finding pivot elements in matrix computations
  - graph coarsening: making the graph smaller by merging similar vertices before partitioning it for parallel computations
  - bioinformatics: finding similarity in Protein-Protein Interaction networks

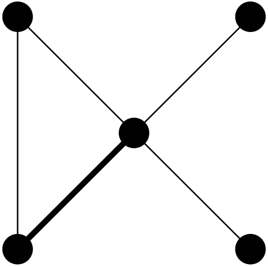
Motivation of greedy/approximation graph matching

- Optimal solution is possible in polynomial time.
- Time for weighted matching in graph $G = (V, E)$ is $\mathcal{O}(mn + n^2 \log n)$ with $n = |V|$ the number of vertices, and $m = |E|$ the number of edges (Gabow 1990).
- The aim is a billion vertices, $n = 10^9$, with 100 edges per vertex, i.e. $m = 10^{11}$.
- Thus, a time of $\mathcal{O}(10^{20}) = 100,000$ Petaflop units is far too long. Fastest supercomputer today, the Tianhe-2, performs 33.8 Petaflop/s.
- We need linear-time greedy or approximation algorithms.

Formal definition of graph matching

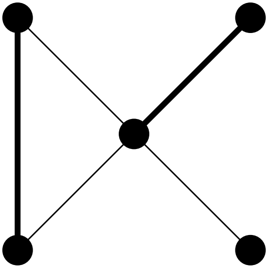
- A graph is a pair $G = (V, E)$ with vertices V and edges E.
- All edges $e \in E$ are of the form $e = (v, w)$ for vertices $v, w \in V$.
- A matching is a collection $M \subseteq E$ of disjoint edges.
- Here, the graph is undirected, so $(v, w) = (w, v)$.

Maximal matching



- A matching is maximal if we cannot enlarge it further by adding another edge to it.

Maximum matching



- A matching is maximum if it possesses the largest possible number of edges, compared to all other matchings.

Edge-weighted matching

- If the edges are provided with weights $\omega : E \to \mathbb{R}_{>0}$, finding a matching M which maximises

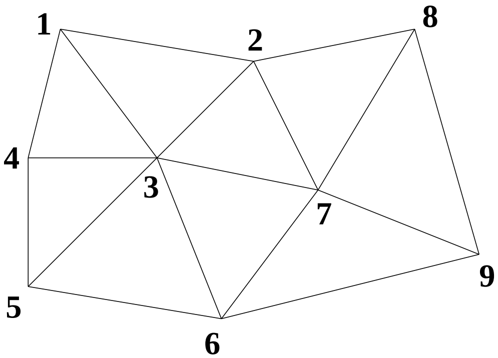
$$\omega(M) = \sum_{e \in M} \omega(e),$$

  is called edge-weighted matching.
- Greedy matching provides us with maximal matchings, but not necessarily with maximum possible weight.

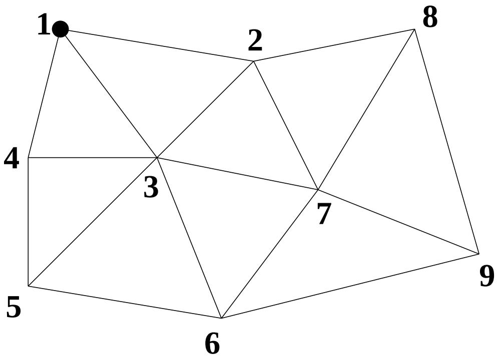
Sequential greedy matching

- In random order, vertices $v \in V$ select and match neighbours one-by-one.
- Here, we can pick
  - the first available neighbour w of v (greedy random matching)
  - the neighbour w with maximum $\omega(v, w)$ (greedy weighted matching)
- Or: we sort the edges by weight, and successively match the vertices v and w of the heaviest available edge (v, w) (greedy matching)

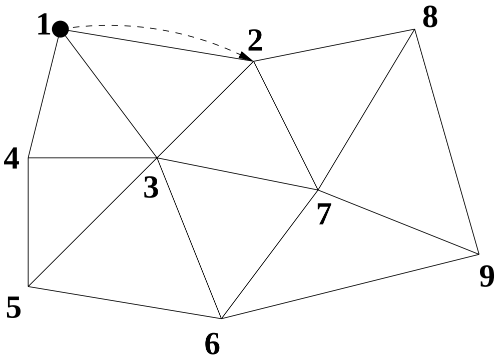
Sequential greedy matching



Sequential greedy matching



Sequential greedy matching



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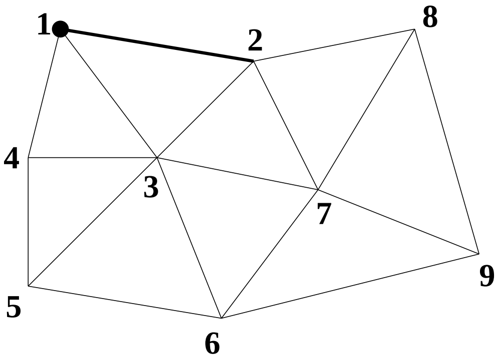
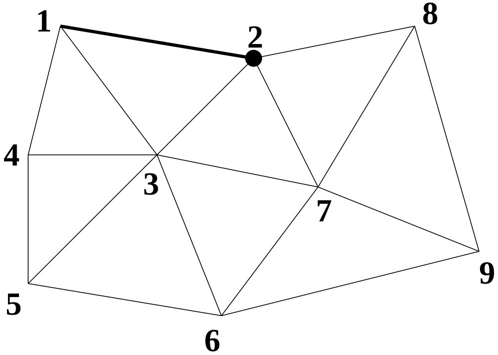
Sequential greedy matching



Sequential greedy matching



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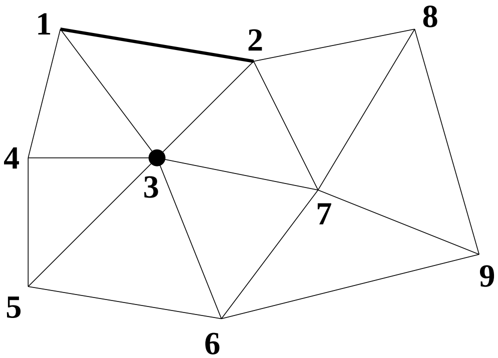
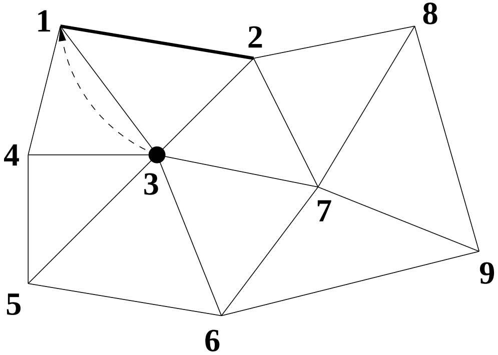
Sequential greedy matching



Sequential greedy matching



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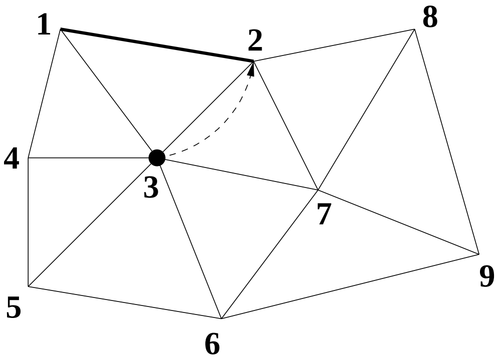
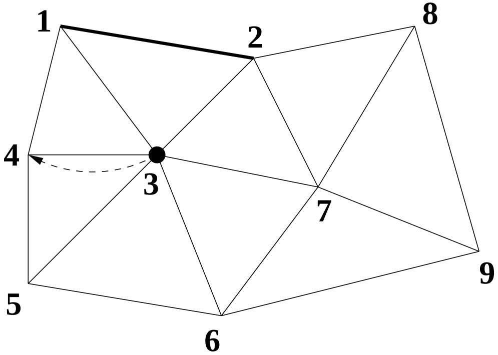
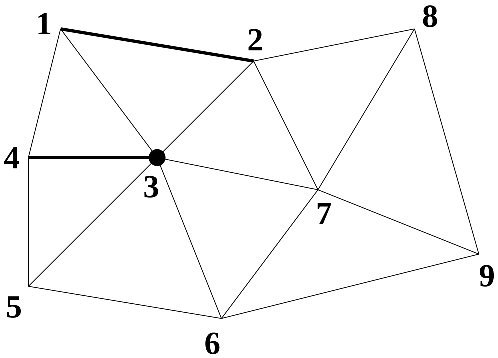


Sequential greedy matching



Sequential greedy matching



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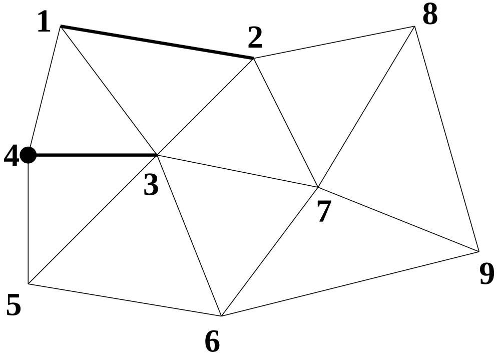
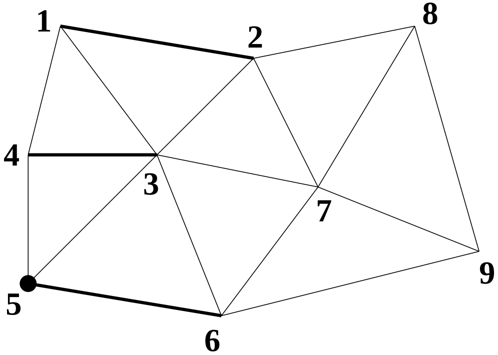
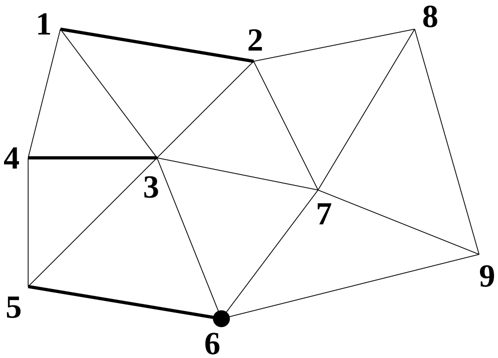
Sequential greedy matching



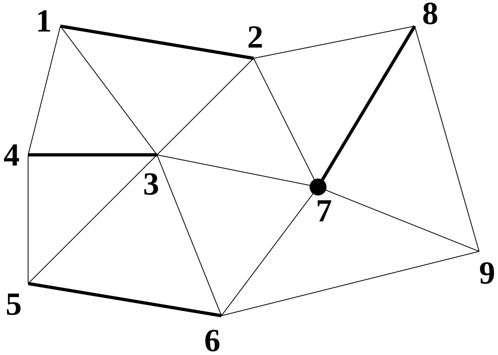
Sequential greedy matching



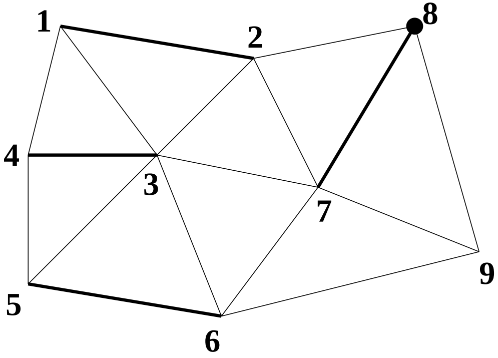
Sequential greedy matching



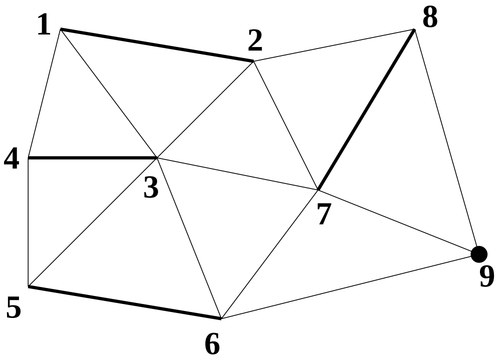
Sequential greedy matching



Sequential greedy matching



Sequential greedy matching



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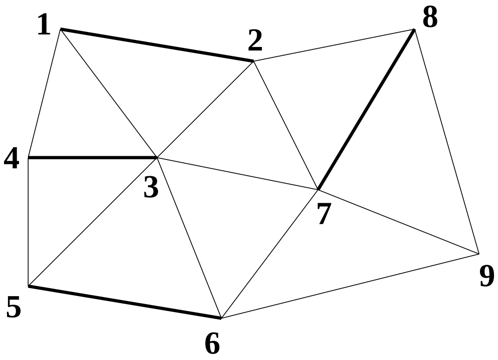
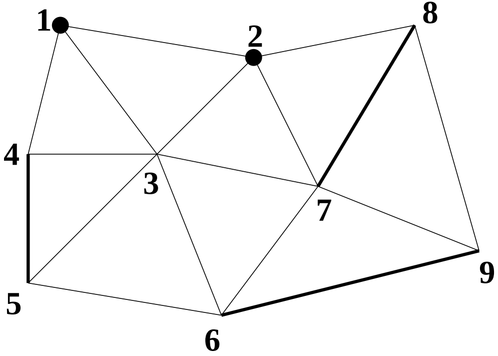
Sequential greedy matching



Greedy is a 1/2-approximation algorithm

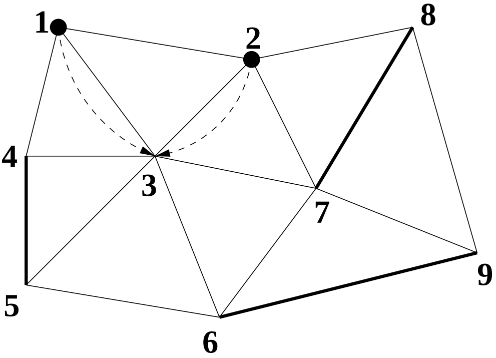
- Weight $\omega(M) \geq \omega_{\text{optimal}}/2$
- Cardinality $|M| \geq |M_{\text{card}-\text{max}}|/2$, because M is maximal.
- Time complexity is $\mathcal{O}(m \log m)$, because all edges must be sorted.

Parallel greedy matching: trouble



Suppose we match vertices simultaneously.

Parallel greedy matching: trouble



Two vertices each find an unmatched neighbour...

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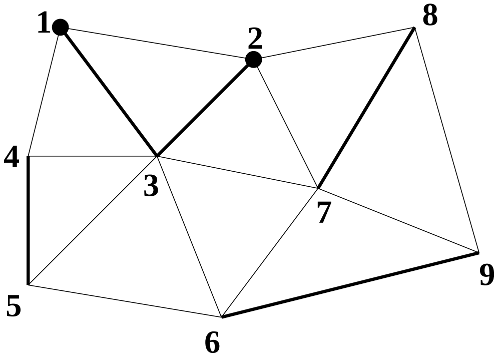
Parallel greedy matching: trouble



. . . but generate an invalid matching.

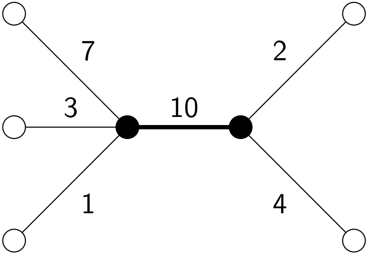
Dominant-edge algorithm

while $E \neq \emptyset$ **do**
  pick dominant edge $(v, w) \in E$
  $M := M \cup \{(v, w)\}$
  $E := E \setminus \{(x, y) \in E : x = v \vee x = w\}$
  $V := V \setminus \{v, w\}$
return M

- An edge $(v, w) \in E$ is dominant if

$$\omega(v, w) = \max\{\omega(x, y) : (x, y) \in E \wedge (x = v \vee x = w)\}$$

Dominant edge



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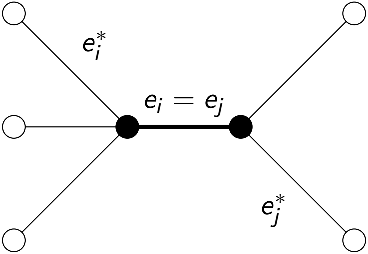
Proof: algorithm is 1/2-approximation

- Let M be the matching produced by the dominant-edge algorithm.
- Let M^* be a maximum matching with weight $\omega_{\rm optimal}$.
- Let $M^* = \{e_0^*, \ldots, e_{k-1}^*\}$. For each edge $e_i^* \in M^*$, define an edge $e_i \in M$, as follows. If $e_i^* \in M$, $e_i = e_i^*$, otherwise e_i is the edge that removes e_i^* from E in the algorithm.
- It may happen that $e_i = e_j$ for $i \neq j$.
- $\omega(e_i) \geq \omega(e_i^*)$ for all i, since e_i is locally dominant in the algorithm and removes e_i^*, or $e_i = e_i^*$.

Proof (cont'd)



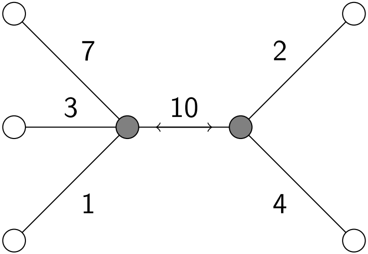
- Every edge $e \in M$ can occur at most twice in the list of e_i's, since it can remove from E at most 2 edges from M^*.
- $$2\omega(M) \geq \sum_{i=0}^{k-1} \omega(e_i) \geq \sum_{i=0}^{k-1} \omega(e_i^*) = \omega_{\text{optimal}}$$
- Hence $\omega(M) \geq \omega_{\text{optimal}}/2$.

Sequential approximation algorithm: initialisation

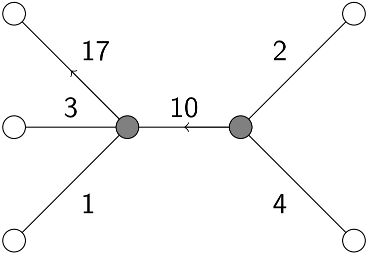
function SeqMatching(V, E)
  for all $v \in V$ **do**
    $pref(v) = null$
  $D := \emptyset$
  $M := \emptyset$

  { Find dominant edges }
  for all $v \in V$ **do**
    $Adj_v := \{w \in V : (v, w) \in E\}$
    $pref(v) := \mathrm{argmax}\{\omega(v, w) \ : \ w \in Adj_v\}$
    if $pref(pref(v)) = v$ **then**
      $D := D \cup \{v, pref(v)\}$
      $M := M \cup \{(v, pref(v))\}$

Mutual preferences



Non-mutual preferences



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Sequential approximation algorithm: main loop

while $D \neq \emptyset$ **do**
  pick $v \in D$
  $D := D \setminus \{v\}$
  for all $x \in Adj_v \setminus \{pref(v)\} \; : \; (x, pref(x)) \notin M$ **do**
    $Adj_x := Adj_x \setminus \{v\}$
    $pref(x) := \operatorname{argmax}\{\omega(x, w) \; : \; w \in Adj_x\}$
    if $pref(pref(x)) = x$ **then**
      $D := D \cup \{x, pref(x)\}$
      $M := M \cup \{(x, pref(x))\}$
return M

Properties of the dominant-edge algorithm

- Dominant-edge algorithm is a 1/2-approximation:

$$\omega(M) \geq \omega_{\mathrm{optimal}}/2$$

- Dominant edge means mutual preference:

$$v = \mathit{pref}(w) \text{ and } w = \mathit{pref}(v).$$

- Dominance is a local property: easy to parallelise.
- Algorithm keeps going until set of dominant vertices D is empty and matching M is maximal.
- Assumption without loss of generality: weights are unique. Otherwise, use vertex numbering to break ties.

Time complexity

- Linear time complexity $\mathcal{O}(|E|)$ if edges of each vertex are sorted by weight.
- Sorting costs are

$$\sum_v deg(v) \log deg(v) \leq \sum_v deg(v) \log \Delta = 2|E| \log \Delta,$$

  where Δ is the maximum vertex degree.
- This algorithm is based on a dominant-edge algorithm by Preis (1999), called LAM, which is linear-time $\mathcal{O}(|E|)$, does not need sorting, and also is a 1/2-approximation, but is hard to parallelise.

Parallel algorithm (Manne & Bisseling, 2007)

- Processor $P(s)$ has vertex set V_s, with

$$\bigcup_{s=0}^{p-1} V_s = V$$

  and $V_s \cap V_t = \emptyset$ if $s \neq t$.
- This is a p-way partitioning of the vertex set.

Halo vertices

- The adjacency set Adj_v of a vertex v may contain vertices w from another processor.
- We define the set of halo vertices

$$H_s = \bigcup_{v \in V_s} Adj_v \setminus V_s$$

- The weights $\omega(v, w)$ are stored with the edges, for all $v \in V_s$ and $w \in V_s \cup H_s$.
- $E_s = \{(v, w) \in E : v \in V_s\}$
  is the subset of all the edges connected to V_s.

Parallel algorithm for $P(s)$: initialisation

function $\textsc{ParMatching}(V_s, H_s, E_s, \text{distribution } \phi)$
  for all $v \in V_s$ **do**
    $\mathit{pref}(v) = \mathit{null}$
  $D_s := \emptyset$
  $M_s := \emptyset$

  { Find dominant edges }
  for all $v \in V_s$ **do**
    $\mathit{Adj}_v := \{w \in V_s \cup H_s : (v, w) \in E_s\}$
    $\mathit{SetNewPreference}(v, \mathit{Adj}_v, \mathit{pref}, V_s, D_s, M_s, \phi)$
  Sync

Setting a vertex preference

function $\textsc{SetNewPreference}(v, Adj, V, D, M, \phi)$

$\quad pref(v) := \text{argmax}\{\omega(v, w) \ : \ w \in Adj\}$

$\quad$ **if** $pref(v) \in V$ **then**

$\quad\quad$ **if** $pref(pref(v)) = v$ **then**

$\quad\quad\quad D := D \cup \{v, pref(v)\}$

$\quad\quad\quad M := M \cup \{(v, pref(v))\}$

$\quad$ **else**

$\quad\quad$ put $proposal(v, pref(v))$ in $P(\phi(pref(v)))$

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How to propose

Source: www.theguardian.com

proposal(v, w): v proposes to w

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Parallel algorithm for $P(s)$: main loop

while $D_s \neq \emptyset$ **do**
    pick $v \in D_s$
    $D_s := D_s \setminus \{v\}$
    for all $x \in Adj_v \setminus \{pref(v)\} \;:\; (x, pref(x)) \notin M_s$ **do**
        if $x \in V_s$ **then**
            $Adj_x := Adj_x \setminus \{v\}$
            $SetNewPreference(x, Adj_x, pref, V_s, D_s, M_s, \phi)$
        else $\{x \in H_s\}$
            put $unavailable(v, x)$ in $P(\phi(x))$
Sync

Parallel algorithm for $P(s)$: communication

for all messages m received **do**
  if $m = \mathit{proposal}(x, y)$ **then**
    if $\mathit{pref}(y) = x$ **then**
      $D_s := D_s \cup \{y\}$
      $M_s := M_s \cup \{(x, y)\}$
      put $\mathit{accepted}(x, y)$ in $P(\phi(x))$
  if $m = \mathit{accepted}(x, y)$ **then**
    $D_s := D_s \cup \{x\}$
    $M_s := M_s \cup \{(x, y)\}$
  if $m = \mathit{unavailable}(v, x)$ **then**
    if $(x, \mathit{pref}(x)) \notin M_s$ **then**
      $\mathit{Adj}_x := \mathit{Adj}_x \setminus \{v\}$
      $\mathit{SetNewPreference}(x, \mathit{Adj}_x, \mathit{pref}, V_s, D_s, M_s, \phi)$

Termination

- The algorithm alternates supersteps of computation running the main loop and communication handling the received messages.
- The whole algorithm can terminate when no messages have been received by processor $P(s)$ and the local set D_s is empty, for all s.
- This can be checked at every synchronisation point.

Load balance

- Processors can have different amounts of work, even if they have the same number of vertices or edges.
- Use can be made of a global clock based on ticks, the unit of time needed to handle a vertex x (in $\mathcal{O}(1)$).
- After every k ticks, everybody synchronises.

Synchronisation frequency

- Guidance for the choice of k is provided by the BSP parameter l, the cost of a global synchronisation.
- Choosing $k \geq l$ guarantees that at most 50% of the total time is spent in synchronisation.
- Choosing k sufficiently small will cause all processors to be busy during most supersteps.
- Good choice: $k = 2l$?

Sending messages

- The BSP system takes care that messages are sent automatically, in bulk. A useful BSPlib primitive for doing this is `bsp_send`.
- In the next superstep, all received messages are read (using `bsp_move`) and processed.
- Google's Pregel system (Malewicz 2010) follows this BSP style.

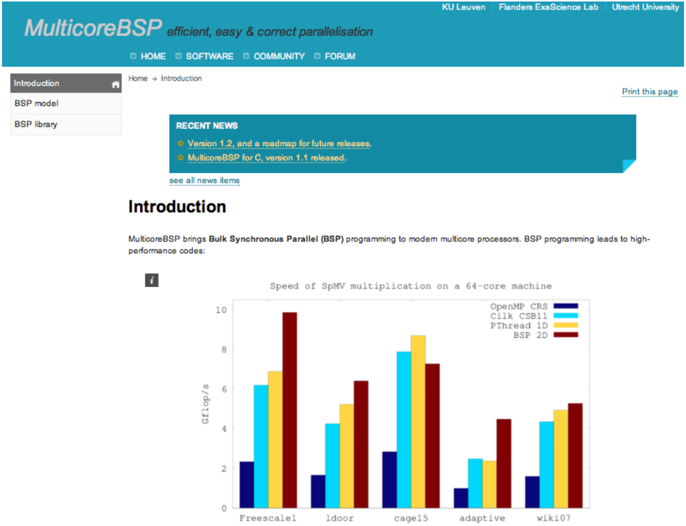
Further improvement: edge-based (2D) distribution

Name	*SpMV* 1*D*	*SpMV* 2*D*	*Matching* 1*D*	*Matching* 2*D*
rw9 (af_shell10)	113	105	169	150
rw10 (boneS10)	150	145	228	189
rw11 (Stanford)	340	141	479	234
rw12 (gupta3)	710	44	1,305	61
rw13 (St_Berk.)	716	448	1,152	812
rw14 (F1)	139	130	148	139
sw1 (small world)	1,007	417	2,111	303
sw2	1,957	829	3,999	563
sw3	2,017	832	4,255	528
er1 (random)	1,856	1,133	1,788	1,157
er2	3,451	1,841	3,721	1,635
er3	5,476	2,569	6,350	1,990

Communication volume in sparse matrix–vector multiplication and Karp–Sipser matching.

Source: Patwary, Bisseling, Manne (2010).

MulticoreBSP enables shared-memory BSP



Albert-Jan Yzelman 2014, www.multicorebsp.org

Matching with MulticoreBSP

- BSP program can remain the same, giving portability.
- To exploit the ease of reading data in shared memory, the `bsp_direct_get` is available in MulticoreBSP.
- This performs the communication immediately and blocks until the communication has been carried out.
- Possible use: replace the set M_s of matched edges by a boolean array $matched_s$ marking the local matched vertices.
- This array can be read by all processors using `bsp_direct_get`, to replace the check $(x, pref(x)) \notin M_s$.

Conclusions and outlook

- BSP is extremely suitable for parallel graph computations:
  - no need to worry about communication because we buffer messages until the next synchronisation;
  - no need for send-receive pairs;
  - BSP cost model gives synchronisation frequency;
  - correctness proof of algorithm becomes simpler;
  - no deadlock possible.